Operator content of the fusion cyclic solid-on-solid model

1993 J. Phys. A: Math. Gen. 26 51

(http://iopscience.iop.org/0305-4470/26/1/010)

J. Phys. A: Math. Gen. 26 (1993) 51–62. Printed in the UK

Operator content of the fusion cyclic solid-on-solid model

Doochul Kim†, Jae Dong Noh† and Paul A Pearce‡

† Department of Physics and Center for Theoretical Physics, Seoul National University, Seoul 151-742, Korea

‡ Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia

Received 27 April 1992

Abstract. The fusion model of the L-state cyclic solid-on-solid (CSOS) model is considered at criticality. Each Z_L-charge sector of the fusion CSOS model is related to a sum of total spin sectors of the fusion vertex model with a seam on which the vertex weights are modified by a phase factor. The latter in turn becomes the higher-spin XXZ quantum chain with a twisted boundary condition in the extremely anisotropic limit. Using the known operator content of the higher-spin XXZ chain, we deduce that of the fusion CSOS model. The modular invariant partition function of the fusion CSOS at fusion level k is a sum of products of the $Z(k)$ parafermionic and free boson sectors with the effective coupling $g'=L^2(1/k-\lambda/\pi)$ for L odd and $g''=L^2(1/k-\lambda/\pi)/4$ for L even, where λ is the crossing parameter. When L is odd and a multiple of k, or when $L/2$ is a multiple of k for L even, the modular invariant partition function becomes a simple product of the $Z(k)$ parafermionic and the Gaussian partition function.

1. Introduction

The L-state cyclic solid-on-solid (CSOS) model [1,2] is a class of integrable lattice models where the spins or heights on each lattice site can take the integer values $0,1,\dots,L-1$ ($L\geqslant 3$), with the height L being identified with 0. The heights of adjacent sites are restricted to differ by ± 1 mod L. Due to this restriction, each allowed configuration of the CSOS model can be mapped to a unique six-vertex arrow configuration but not vice versa. The allowed face configurations and their corresponding assignment of six-vertex arrow configurations are shown in figure 1. The model is parametrized by L and the crossing parameter λ, which can take a set of discrete values $\lambda=\pi s/L$, with $s=1,2,\dots,L-1$ co-prime to L and is solvable in a three-dimensional manifold.

At the critical point, non-vanishing face weights of the CSOS model reduce to those of the corresponding six-vertex model. They are

$$\begin{aligned} W_{1,1}(a,a+1,a,a-1|u) &= W_{1,1}(a,a-1,a,a+1|u)=s_{-1}\\ W_{1,1}(a+1,a,a-1,a|u) &= W_{1,1}(a-1,a,a+1,a|u)=s_0 \\ W_{1,1}(a+1,a,a+1,a|u) &= W_{1,1}(a-1,a,a-1,a|u)=1 \end{aligned}\tag{1}$$

where $W_{1,1}(a,b,c,d|u)$ is the weight for a face with corner heights $\{a,b,c,d\}$ going counterclockwise starting from the lower left corner, and

$$s_n\equiv-\frac{\sin(u+n\lambda)}{\sin(\lambda)}\,.\tag{2}$$

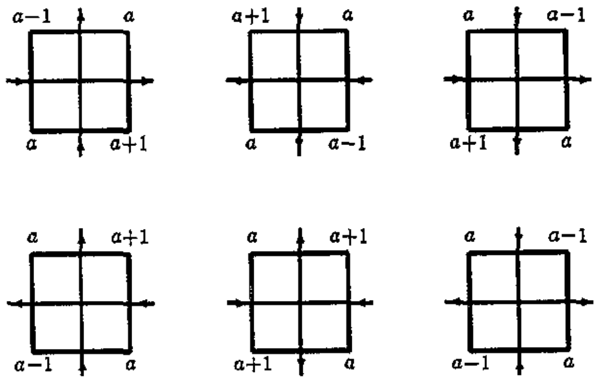


Figure 1. Non-vanishing face weights of the CSOS model and assignments of vertex configurations. At criticality, weights of the CSOS model become the same as the corresponding vertex weights of the six-vertex model.

The subscript $(1,1)$ stands for the fusion level to be discussed below. We use the particular gauge in which the third and fourth weights of (1) are negative for $0 < u < \lambda$. Here, u is the spectral parameter controlling the spatial anisotropy of the interactions. With this parametrization, the Yang–Baxter equation is satisfied for any λ and it is not necessary to restrict it to a set of discrete values any more; we take it to be a real number in the range $0 < \lambda < \pi$. In [3], the operator content of the CSOS model on torus with general toroidal boundary conditions is derived using its relations to the six-vertex model and the known operator content of the latter. In particular, the modular invariant partition function (MIPF) for even numbers of rows and columns is found to be exactly the Coulombic (or Gaussian) partition function whose coupling constant g' (g'') for L odd (even) (see for example [4–6], is related to the CSOS model parameters by†

$$g' = L^2(1 - \lambda/\pi) \qquad \text{if } L \text{ is odd}$$
$$g'' = L^2(1 - \lambda/\pi)/4 \qquad \text{if } L \text{ is even}.$$

The k-fusion is a procedure in which $k \times k$ blocks of vertices or faces are put together and partially traced to obtain new integrable weights [7–9]. The 2-fusion of the six-vertex model for example leads, after a gauge transformation, to the 19-vertex model [4, 10]. The vertex model obtained by k-fusion of the six-vertex model will be called the k-fusion vertex model. Its MIPF is derived in [4] and is a sum of tensor products of the $Z(k)$ parafermionic and free bosonic sectors (see below). In the extremely anisotropic limit $u \to 0$, the model reduces to the higher-spin XXZ quantum chain [11, 12]; that is, the logarithmic derivative at $u = 0$ of the row transfer matrix of k-fusion vertex model is the spin-$k/2$ XXZ quantum chain Hamiltonian. The associated central charge is

$$c = 3k/(k+2)\,. \tag{3}$$

† We take this opportunity to correct an error in [3]. The term $+\pi i m/N$ should be added to the right-hand side of equation (26) of [3]. This gives the crucial sign term $(-1)^m$ in the modular covariant partition function for a lattice with an odd number of rows. We thank to J-Y Choi for pointing out the missing term.

Fusion of the RSOS model [8,9] leads to the class of models whose criticality in the continuum limit is described by conformal field theory associated with the co-set construction based on SU(2) [13]. Recently, fusion of the CSOS model has been studied in [14] and [15].

In this work, we consider the critical fusion CSOS model obtained by the k-fusion procedure of the critical CSOS model. In particular, we focus on the operator content and derive its MIPF. Since the critical CSOS weights can be mapped to those of the integrable six-vertex model, properties of the critical k-fusion CSOS model can be inferred from those of the k-fusion vertex model as discussed below. The essential ingredient in this approach is the full operator content of the k-fusion vertex model with a twisted boundary condition. Its exact form has been conjectured by Alcaraz and Martins [16] from numerical results of XXZ quantum chain for $k = 2$ and 3. (The leading term, i.e. the central charge for $k = 2$ is calculated analytically in [10].) We combine this and the result of [4] to derive the MIPF for the k-fusion CSOS model.

2. Row-transfer matrix structure

In this section, we discuss the relation between the row-transfer matrix (RTM) of the critical fusion CSOS model and that of the fusion vertex model. First we consider the RTM of the L-state critical CSOS model whose face weights are given in (1) [3]. We put the system on the square lattice with N columns and M rows and impose the periodic boundary conditions in both directions. For simplicity, we confine our discussions to the case where N and M are both even. We discuss briefly the cases of other shift boundary conditions and odd parities of N and M in the last section. A height configuration of a row $\boldsymbol{a} = \{a_1, a_2, \ldots, a_N\}$ is allowed or admissible if each pair $(a_i,\ a_{i+1})$ satisfies the adjacency condition. An admissible configuration can then be represented by $(a_1,\ \boldsymbol{\sigma})$ where $\boldsymbol{\sigma} = \{\sigma_1, \ldots, \sigma_N\}$ and

$$\sigma_i = a_{i+1} - a_i \qquad \bmod L\,. \tag{4}$$

With the association of $\sigma_i = 1$ (-1) to the up (down) arrow on the corresponding vertical bond of the dual lattice, $\boldsymbol{\sigma}$ stands for an arrow configuration of a row of vertical bonds. The 'total spin' defined by

$$Q = \frac{1}{2}\sum_{i=1}^{N} \sigma_i \tag{5}$$

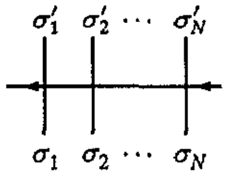
is a conserved quantity from row to row because of the local arrow conservations and one can consider each Q sector separately. Q takes integer values between $-N/2$ and $N/2$ but the periodic boundary condition for the L-state CSOS model further restricts possible values of $2Q$ to 0 mod L. Thus Q takes the values of the set

$$R_L \boxminus \begin{cases} LZ & \text{for } L \text{ odd} \\ (L/2)Z & \text{for } L \text{ even} \end{cases} \tag{6}$$

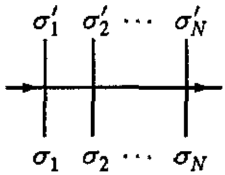
with implicit understanding of $|Q| \leqslant N/2$ before the thermodynamic limit.

Let $V^{(1,1)}_{\mu,Q}$ ($\mu = \pm 1$, $Q = 0, \pm 1, \ldots, \pm N/2$) be the six-vertex RTM in the sector Q with the first horizontal arrow fixed to left (right) when $\mu = 1$ (-1). This can be written graphically as

$V_{1,Q}^{(1,1)}(\sigma \mid \sigma') =$



$V_{-1,Q}^{(1,1)}(\sigma \mid \sigma') =$



with $\sum \sigma_i = \sum \sigma_i' = 2Q$. $V_{\mu,Q}^{(1,1)}$ with arguments stands for its matrix element. The superscript $(1,1)$ is added to distinguish various fusion levels and u and λ dependences are implicit. The standard RTM T_1 of the six-vertex model [17] then can be written as

$$T_1 = \bigoplus_Q \{V_{1,Q}^{(1,1)} + V_{-1,Q}^{(1,1)}\} \tag{7}$$

where $\bigoplus$ denotes direct sum and the sum is over $Q = 0, \pm 1, \ldots, \pm N/2$. The logarithmic derivative of T_1 at $u = 0$ gives the spin-$\frac{1}{2}$ XXZ chain Hamiltonian. The twisted boundary condition on the XXZ chain is obtained by choosing

$$S_{N+1}^{\pm} = \mathrm{e}^{\pm \mathrm{i}\phi} S_1^{\pm} \tag{8}$$

where $S_i^{\pm}$ is the spin raising/lowering operator for spin $S = \frac{1}{2}$, and ϕ is the twist angle. It has been of considerable interest since the operator content of other $c < 1$ theories can be generated by appropriate choice of sectors and the value of ϕ [18]. In the vertex-model language, it is equivalent to the six-vertex model with a seam in which the vertex weights are modified by a phase factor $\exp(\mathrm{i}\mu\phi/2)$ with $\mu = \pm 1$ depending on the horizontal arrow direction of the seam [3]. Its RTM T_1' takes the form

$$T_1' = \bigoplus_Q \left\{\exp(\mathrm{i}\phi/2) V_{1,Q}^{(1,1)} + \exp(-\mathrm{i}\phi/2) V_{-1,Q}^{(1,1)}\right\} . \tag{9}$$

The RTM V_1 of the L-state CSOS model with the states labelled as (a_1, σ), etc, takes the tridiagonal block structure

$$V_1(a_1, \sigma | a_1', \sigma') = \begin{cases} \bigoplus_{Q \in R_L} V_{\mu,Q}^{(1,1)}(\sigma|\sigma') & \text{if } a_1' - a_1 = \mu \bmod L \\ 0 & \text{otherwise}. \end{cases} \tag{10}$$

Thanks to the cyclic symmetry

$$V_1(a_1, \sigma | a_1', \sigma') = V_1(a_1 + 1, \sigma | a_1' + 1, \sigma') \tag{11}$$

we can block-diagonalize V_1 into the 'Z_L-charge' sectors with the Z_L-charge P taking the values of the set

$$S_L = \{0, 1, \dots, L-1\}\,. \tag{12}$$

Therefore we have

$$V_1 = \bigoplus_{P\in S_L} \bigoplus_{Q\in R_L} \left\{ \exp(2\pi i P/L) V_{1,Q}^{(1,1)} + \exp(-2\pi i P/L) V_{-1,Q}^{(1,1)} \right\}\,. \tag{13}$$

Now we introduce the k-fusion models of the CSOS model. Define, as in [9],

$$W'_{k,k}(a,b,c,d|u) = \sum \prod_{0\leqslant i,j\leqslant k-1} W_{1,1}(\alpha_{i,j}, \alpha_{i+1,j}, \alpha_{i+1,j+1}, \alpha_{i,j+1}|u + (i+j+1-k)\lambda) \tag{14}$$

with $\alpha_{0,0} = a$, $\alpha_{k,0} = b$, $\alpha_{k,k} = c$, $\alpha_{0,k} = d$, where the sum is taken over all allowed configurations of $\{\alpha_{i,j}\}$, keeping fixed the corner heights a, b, c, d, and the right/top boundary heights. This is pictorially shown for $k = 3$ in figure 2.

When the fusion level k satisfies the condition

$$\begin{aligned} &k \leqslant L-1 \qquad \text{for } L \text{ odd} \\ &k \leqslant L/2-1 \qquad \text{for } L \text{ even} \end{aligned} \tag{15}$$

one can show, in the manner of [9], that the resulting sum in (14) is independent of the remaining boundary heights (open circles in figure 2) and is factorized by the factor $f_0 f_{-1} \cdots f_{-k+1}$ where

$$f_a = s_a s_{a+1} \cdots s_{a+k-2}\,. \tag{16}$$

The face weights of the k-fusion CSOS model at criticality are then

$$W_{k,k}(a,b,c,d|u) = W'_{k,k}(a,b,c,d|u) \Big/ \prod_{p=0}^{k-1} f_{-p}\,. \tag{17}$$

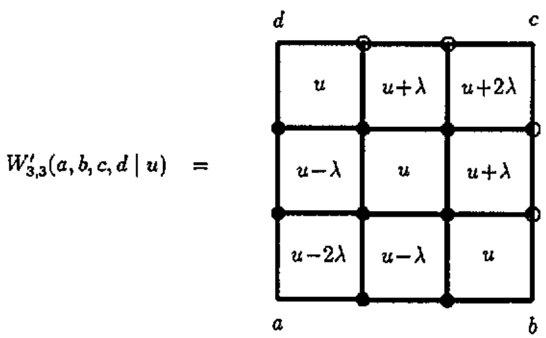


Figure 2. Graphical representation of the (3×3)-fusion. The heights on sites with full circles are summed over while those on the right and top boundary heights as well as that on the bottom left corner are kept constant.

An allowed row configuration $\{a_1, a_2, \ldots, a_N\}$ can then be represented by (a_1, σ) as before but now each σ_i takes the values

$$\sigma_i = a_{i+1} - a_i \qquad \bmod L = k, k-2, \ldots, -k \,. \tag{18}$$

Equation (17) also defines the vertex weights of the k-fusion vertex model where the four bond states are $\{b-a,\ c-b,\ c-d,\ d-a\}$ anticlockwise starting from the lower vertical one.

Similarly to the $k = 1$ case, we define $V_{\mu,Q}^{(k,k)}$ with $k \geqslant 1$, $\mu = k, k-2, \ldots, -k$ and $Q = 0, \pm 1, \ldots, \pm kN/2$, as the k-fusion vertex model RTM in the total spin sector Q with the first horizontal bond state fixed to μ. In the fusion vertex model with a seam, the vertex weights along the seam pick up the phase factor $\exp(\mathrm{i}\mu\phi/2)$, with μ now denoting the horizontal bond state ($\mu = k, k-2, \ldots, -k$). Then the RTM T'_k of the k-fusion model with the seam on the first column is given as

$$T'_k = \bigoplus_Q \sum_\mu \exp(\mathrm{i}\mu\phi/2) V_{\mu,Q}^{(k,k)} \,. \tag{19}$$

In the higher-spin XXZ chain limit, the effect of the seam translates into the twisted boundary condition (8) with $S = k/2$.

As to the L-state k-fusion CSOS model, the RTM V_k maintains the cyclic $L \times L$ block structure as in (10). After block-diagonalizing into the Z_L-charge sectors, one can write it as

$$V_k = \bigoplus_{P \in S_L} \bigoplus_{Q \in R_L} \sum_\mu \exp(2\pi \mathrm{i}\mu P/L) V_{\mu,Q}^{(k,k)} \,. \tag{20}$$

Comparing (19) with (20), one sees that the operator content of the latter can be inferred from that of the former. When k is out of the range (15), one cannot define local face weights for CSOS model due to the winding configurations [15]. However one can still define the fusion models by that whose RTM is given by (20) for arbitrary k.

3. Operator content of the fusion CSOS model

Conformal invariance of a critical statistical system implies that its partition function on a torus is determined, apart from the non-universal bulk contribution, by its underlying conformal field theory and is a universal function of the modular parameter τ in the continuum limit. We are interested in this MIPF of the fusion CSOS model in this section. For the six-vertex model on an $N \times M$ lattice, τ is related to the spectral parameter by [19]

$$\tau = \frac{M}{N} \exp[\mathrm{i}\pi(1 - u/\lambda)] \,. \tag{21}$$

This relation should also be valid for fusion vertex models.

Di Francesco *et al* [4] have argued that the MIPF of the fusion vertex model is a sum of tensor products of free bosonic and $Z(k)$-parafermionic sectors. Some of their results were checked numerically. It was later confirmed by Alcaraz and Martins

[20] in a more extensive numerical work on the Bethe ansatz equation for the XXZ chain version. We recall that the Coulombic or Gaussian partition function with the coupling constant g is given by [4]

$$Z_{1-c}(g) = \sum_{m,m' \in \mathbf{Z}} Z_{m,m'}(g) \tag{22}$$

where

$$Z_{m,m'}(g) = \sqrt{\frac{g}{\tau''}} \frac{1}{|\eta(q)|^2} \exp\left(-\pi g |m' - m\tau|^2/\tau''\right) \tag{23}$$

$\tau = \tau' + \mathrm{i}\tau''$, $q = \exp(2\pi\mathrm{i}\tau)$ and $\eta(q)$ is the Dedekind function

$$\eta = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \, . \tag{24}$$

After the Poisson transformation, it takes the form

$$Z_{1-c}(g) = \frac{1}{|\eta|^2} \sum_{m,n \in \mathbf{Z}} q^{\Delta_{m,n}} \bar{q}^{\bar{\Delta}_{m,n}} \tag{25}$$

where $\bar{q}$ is the complex conjugate of q and the conformal dimensions ($\Delta_{m,n}(g)$, $\bar{\Delta}_{m,n}(g)$) in the free bosonic sectors (m, n) are given by

$$\Delta_{m,n}(g) = \left(n\sqrt{g} + m/\sqrt{g}\right)^2/4 \tag{26}$$

$$\bar{\Delta}_{m,n}(g) = \left(n\sqrt{g} - m/\sqrt{g}\right)^2/4 \, . \tag{27}$$

The $Z(k)$-parafermionic field theory is first studied in [21]. In this model, one can impose twisted boundary conditions $\exp(2\pi\mathrm{i}r/k)$ and $\exp(2\pi\mathrm{i}s/k)$ on the spin variables along the space and time directions, respectively. The respective toroidal partition functions $\mathcal{Z}_k(r,s)$ are derived in [22]. We define $\tilde{\mathcal{Z}}_k(r,s)$ as the partial trace over the Z_k-charge sector s with the spatial boundary condition $\exp(2\pi\mathrm{i}r/k)$. $\mathcal{Z}_k$ is related to $\tilde{\mathcal{Z}}_k$ by

$$\mathcal{Z}_k(r,s) = \sum_{s'=0}^{k-1} \exp(2\pi\mathrm{i}ss'/k) \tilde{\mathcal{Z}}_k(r,s') \tag{28}$$

and $\tilde{\mathcal{Z}}_k$ in turn is related to the string function c_m^ℓ (see equation (3.33) of [22]) via

$$\tilde{\mathcal{Z}}_k(r,s) = |\eta|^2 \sum_{\ell=0}^{k} c_{r+s}^\ell \bar{c}_{r-s}^\ell \, . \tag{29}$$

A useful representation of c_m^ℓ is given by [23]

$$\begin{aligned} c_m^\ell = {} & q^{\ell(\ell+2)/4(k+2) - m^2/4k + 1/4(k+2)} \eta^{-3} \sum_{r,s=0}^{\infty} (-1)^{r+s} q^{r(r+1)/2 + s(s+1)/2 + rs(k+1)} \\ & \times \left\{ q^{r(\ell+m)/2 + s(\ell-m)/2} - q^{k+1-\ell+r[k+1-(\ell+m)/2] + s[k+1-(\ell-m)/2]} \right\} \end{aligned} \tag{30}$$

when $\ell - m \in 2Z$ and $c^{\ell}_m = 0$ otherwise. When $k = 2$ and 3, c^{ℓ}_m is the sum of Virasoro characters for $c = \frac{1}{2}$ and $\frac{4}{5}$ minimal theory, respectively. For general k, its leading term is $c^{\ell}_m \sim q^{-c/24+h^{\ell}_m}$ where $c = 3k/(k+2)$ and

$$h^{\ell}_m = \frac{\ell(\ell+2)}{4(k+2)} - \frac{m^2}{4k} \tag{31}$$

for $|m| \leqslant \ell$. For other values of m, the symmetries

$$c^{\ell}_m = c^{\ell}_{-m} = c^{\ell}_{m+2k} = c^{k-\ell}_{k-m} \tag{32}$$

can be used to put m in the indicated range. Since the right-hand side of (29) is periodic in r and s with period k, $\tilde{\mathcal{Z}}_k(r,s)$ and $\mathcal{Z}_k(r,s)$ can be considered to be defined for all r, $s \in Z$ with the same periodicity.

Using these notations, the result of [4] for the MIPF of the fusion vertex model (equation (4.7) of [4]) is given by

$$Z_{k-c}(g) = \sum_{n,m' \in \mathbf{Z}} Z_{n,m'}(g)\mathcal{Z}_k(n,m')\frac{1}{|\eta|^2}\sum_{m,n\in\mathbf{Z}} q^{\Delta_{m/k,n}}\bar{q}^{\bar{\Delta}_{m/k,n}}\tilde{\mathcal{Z}}_k(n,m) \tag{33}$$

where the coupling constant g which enters through the definitions of conformal dimensions $(\Delta, \bar{\Delta})$ is related to the crossing parameter by

$$g = 1/k - \lambda/\pi\,. \tag{34}$$

The two expressions are related by the Poisson transformation. This result is valid for

$$0 < \lambda < \pi/k \tag{35}$$

and we restrict the range of λ accordingly. The summation index n in (33) is identified with the total spin quantum number Q by Alcaraz and Martins [20]. They also studied in [16] the operator content of higher-spin XXZ chain with the twisted boundary condition (8). It was found that the effect of the twisted boundary condition enters through the shift in the subscript of $(\Delta, \bar{\Delta})$ in (33):

$$\Delta_{m/k,n} \rightarrow \Delta_{m/k+\phi/2\pi,n} \tag{36}$$

$$\bar{\Delta}_{m/k,n} \rightarrow \bar{\Delta}_{m/k+\phi/2\pi,n}\,. \tag{37}$$

The partition function in the total spin sector Q of the fusion vertex model with seam is defined by

$$Z_Q(\phi) = \mathrm{tr}\left\{\sum_{\mu}\exp(\mathrm{i}\mu\phi/2)V^{(k,k)}_{\mu,Q}\right\}^M \tag{38}$$

where M is the number of row in the lattice and tr is the trace over $(k+1)^N$ bond states of a row with N columns. Results of [4] and [16] implies that

$$Z_Q(\phi) = \frac{1}{|\eta|^2}\sum_{m\in Z} q^{\Delta_{m/k+\phi/2\pi,Q}}\bar{q}^{\bar{\Delta}_{m/k+\phi/2\pi,Q}}\tilde{\mathcal{Z}}_k(Q,m) \tag{39}$$

where here and below we omit the bulk free energy contribution from the partition function. Combining these results with discussions of the previous section, we deduce that the MIPF of the k-fusion CSOS denoted by $Z_{k-\text{CSOS}}$ is given by

$$Z_{k\text{-CSOS}} = \sum_{P\in S_L} \sum_{Q\in R_L} Z_Q(\phi = 4\pi P/L) . \tag{40}$$

This is our principal result. From this one can read off the operator content for each of P and Q sectors of the fusion CSOS model. For example, the conformal dimensions of the operators in the P, Q sector are

$$\left(\Delta_{m/k+2P/L,Q} + h^{\ell}_{r+s},\ \bar{\Delta}_{m/k+2P/L,Q} + h^{\ell}_{r-s}\right)$$

where $r = Q \bmod k$, $s = m \bmod k$, $m \in Z$ and $0 \leqslant \ell \leqslant k$ with $\ell + r + s$ even. $\Delta_{m,n}$, $\bar{\Delta}_{m,n}$ and h^{ℓ}_m are given by (26), (27) and (31), respectively. The central charge and the composite character of the operators are the same as in the fusion vertex model.

4. Discussions

Equation (40) can be brought into more transparent form by summing over P. Let us first consider the L-odd case. When L is odd, $Q = nL$ with $n \in \mathbf{Z}$. Define $\Delta'_{m,n}$ and $\bar{\Delta}'_{m,n}$ by

$$\Delta'_{m,n} = \Delta_{m/L,nL} \tag{41}$$

$$\bar{\Delta}'_{m,n} = \bar{\Delta}_{m/L,nL} . \tag{42}$$

These are the Gaussian conformal dimensions in the sector (m, n) with the scaled coupling constant

$$g' = L^2 g = L^2(1/k - \lambda/\pi) . \tag{43}$$

Equation (40) can now be written as

$$Z_{k\text{-CSOS}} = \frac{1}{|\eta|^2} \sum_{P\in S_L} \sum_{m,n\in Z} q^{\Delta'_{mL/k+2P,n}}\, \bar{q}^{\bar{\Delta}'_{mL/k+2P,n}} \tilde{Z}_k(nL, m) . \tag{44}$$

Changing the summation index m to $m'k + s$ ($m' \in \mathbf{Z}$, $s = 0, \ldots, k-1$) and using the fact that the set

$$\{m'L + 2P | m' \in Z, P \in S_L\} \tag{45}$$

is equal to $\mathbf{Z}$ for L odd, we can write $Z_{k-\text{CSOS}}$ as

$$Z_{k\text{-CSOS}} = \frac{1}{|\eta|^2} \sum_{s=0}^{k-1} \sum_{m,n\in Z} q^{\Delta'_{m+sL/k,n}}\, \bar{q}^{\bar{\Delta}'_{m+sL/k,n}} \tilde{Z}_k(nL, s) \qquad (L \text{ odd}) . \tag{46}$$

This expression simplifies drastically when L/k is an integer. In this case, equation (46) becomes

$$Z_{k\text{-CSOS}} = \frac{1}{|\eta|^2}\sum_{s=0}^{k-1}\sum_{m,n\in Z} q^{\Delta'_{m,n}}\bar{q}^{\bar{\Delta}'_{m,n}}\tilde{\mathcal{Z}}_k(0,s) = Z_{1\text{-c}}(g')\mathcal{Z}_k(0,0)$$

$$(L \text{ odd}, L{=}0 \bmod k) \tag{47}$$

where we have used (28) and (25). That is, the model becomes the simple direct product of the $Z(k)$-parafermionic model and the Gaussian model. When L/k is not an integer, such simple factorization does not occur. However, a Poisson transformation on (46) leads to a simpler form where the modular invariance is explicitly manifested:

$$Z_{k\text{-CSOS}} = \sum_{n,m\in Z} Z_{n,m}(g')\mathcal{Z}_k(nL, mL) \qquad (L \text{ odd}). \tag{48}$$

When L is even, one is led to define $\Delta''_{m,n}$ and $\bar{\Delta}''_{m,n}$ by

$$\Delta''_{m,n} = \Delta_{2m/L,nL/2} \tag{49}$$

$$\bar{\Delta}''_{m,n} = \bar{\Delta}_{2m/L,nL/2}\,. \tag{50}$$

These are the Gaussian conformal dimensions in the sector (m,n) with the scaled coupling constant

$$g'' = (L/2)^2 g = L^2(1/k - \lambda/\pi)/4\,. \tag{51}$$

Proceeding as before one obtains

$$Z_{k\text{-CSOS}} = \frac{2}{|\eta|^2}\sum_{s=0}^{k-1}\sum_{m,n\in Z} q^{\Delta''_{m+sL/2k,n}}\,\bar{q}^{\bar{\Delta}''_{m+sL/2k,n}}\tilde{\mathcal{Z}}_k(nL/2, s)$$

$$= 2\sum_{n,m\in Z} Z_{n,m}(g'')\mathcal{Z}_k(nL/2, mL/2) \qquad (L \text{ even}). \tag{52}$$

The front factor 2 in (52) arises since the set (45) covers even integers twice for L even. Apart from it, the MIPF for L even is exactly the same as that for L odd as long as we use the effective number of heights $L' = L/2$ instead of L in the expression of the latter. (The origin of this reduction in the effective number of heights is discussed in [3].) In particular when $L/2$ is divided by k, one has

$$Z_{k\text{-CSOS}} = 2Z_{1\text{-C}}(g'')\,\mathcal{Z}_k(0,0) \qquad (L \text{ even}, L/2 = 0 \bmod k)\,. \tag{53}$$

We have considered in detail the cases where N and M are both even and the boundary conditions are periodic in both directions. We give brief discussions for other cases. For the shifted boundary conditions (ℓ, ℓ') [3], the set R_L in (6) has to be generalized to

$$R'_L = \begin{cases} L\mathbf{Z} + \ell/2 & \text{for } L \text{ odd and } \ell + kN \text{ even} \\ L\mathbf{Z} + (\ell + L)/2 & \text{for } L \text{ odd and } \ell + kN \text{ odd} \\ (L/2)\mathbf{Z} + \ell/2 & \text{for } L \text{ even and } \ell + kN \text{ even}. \end{cases} \tag{54}$$

and each Z_L-charge P sector contributes to the partition function with a phase factor $\exp(2\pi i P\ell'/L)$. When N and M are not both even, let

$$\mu = N \bmod 2 \qquad \text{and} \qquad \nu = M \bmod 2\,. \tag{55}$$

For arbitrary μ, $\nu = 0, 1$, (33) is modified as [4]

$$\begin{aligned} Z^{\mu,\nu}_{k-c}(g) &= \sum_{n,m'\in Z} Z_{n+k\mu/2,m'+k\nu/2}(g)\, \mathcal{Z}_k(n,m') \\ &= \frac{1}{|\eta|^2} \sum_{m,n\in Z} (-1)^{m\nu} q^{\Delta_{m/k,n+k\mu/2}} \bar{q}^{\bar{\Delta}_{m/k,n+k\mu/2}} \tilde{\mathcal{Z}}_k(n,m)\,. \end{aligned} \tag{56}$$

Note that this result of [4] is different from that of [20]. We have checked numerically for $N \leqslant 9$ and $k = 2$ that (56) is correct and that the values of $n + k\mu/2$ in (56) correspond to the total spin Q. Accordingly, we conjecture that (39) takes the generalized form

$$Z^{\mu,\nu}_Q(\phi) = \frac{1}{|\eta|^2} \sum_{m\in \mathbf{Z}} (-1)^{m\nu} q^{\Delta_{m/k+\phi/2\pi,Q}} \bar{q}^{\bar{\Delta}_{m/k+\phi/2\pi,Q}} \tilde{\mathcal{Z}}_k(Q + k\mu/2, m) \tag{57}$$

and the modular covariant partition function of the k-fusion CSOS is given by

$$Z^{\mu,\nu,\ell,\ell'}_{k-\mathrm{CSOS}} = \sum_{P\in S_L} \exp(2\pi i P\ell'/L) \sum_{Q\in R'_L} Z^{\mu,\nu}_Q(\phi = 4\pi P/L)\,. \tag{58}$$

In summary, the operator content of the fusion CSOS is obtained for general L, the number of heights, and k, the fusion level. The operator content of each Z_L-charge P and total spin Q sectors is related to that of a corresponding sector of the fusion vertex model with twisted boundary conditions. The MIPF which is obtained by summing over all the sectors is expressed as a tensor product of $Z(k)$-parafermionic and free bosonic sectors. When L is odd and a multiple of k, the MIPF factorizes into the parafermionic and the Gaussian partition function with the effective coupling constant given by (43). When L is even, the effective number of heights is reduced to $L/2$.

Acknowledgments

This work was initiated when one us (DK) visited the Department of Mathematics, University of Melbourne, supported by an ARC grant. It is also supported by the KOSEF grant 901–0203–021–2.

References

[1] Pearce P A and Seaton K A 1988 *Phys. Rev. Lett.* **60** 1347; 1989 *Ann. Phys.* **193** 326
[2] Kuniba A and Yajima T 1988 *J. Stat. Phys.* **52** 829
[3] Kim D and Pearce P A 1989 *J. Phys. A: Math. Gen.* **22** 1439
[4] Di Francesco P, Saleur H and Zuber J-B 1988 *Nucl. Phys.* B **300** 393

[5] Pasquier V 1987 *J. Phys. A: Math. Gen.* **20** L1229
[6] Di Francesco P, Saleur H and Zuber J-B 1987 *Nucl. Phys.* B **285** 454
[7] Kulish P P, Reshetikhin N Yu and Sklyanin E K 1981 *Lett. Math. Phys.* **5** 393
[8] Date E, Jimbo M, Miwa T and Okado M 1986 *Lett. Math. Phys.* **12** 209; 1987 *Lett. Math. Phys.* **14** 97
[9] Date E, Jimbo M, Kuniba A, Miwa T and Okado M 1988 *Adv. Stud. Pure Math.* **16** 17
[10] Klümper A, Batchelor M T and Pearce P A *J. Phys. A: Math. Gen.* 1991 **24** 3111
[11] Babujian H M 1983 *Nucl. Phys.* B **215** 317
[12] Kirillov A N and Reshetikhin N Yu 1987 *J. Phys. A: Math. Gen.* **20** 1565; 1587
[13] Bazhanov V V and Reshetikhin N Yu 1989 *Int. J. Mod. Phys.* A **4** 115
[14] Couch M and Pearce P A 1989 unpublished
[15] Takagi T, Deguchi T and Wadati M *Preprint* University of Tokyo
[16] Alcaraz F C and Martins M J 1990 *J. Phys. A: Math. Gen.* **23** 1439
[17] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
[18] Alcaraz F C, Grimm U and Rittenberg V 1989 *Nucl. Phys.* B **316** 735
[19] Kim D and Pearce P A 1987 *J. Phys. A: Math. Gen.* **20** L451
[20] Alcaraz F C and Martins M J 1988 *Phys. Rev. Lett.* **61** 1529; 1989 *J. Phys. A: Math. Gen.* **22** 1829
[21] Zamolodchikov A B and Fateev V A 1985 *Sov. Phys.-JETP* **62** 215
[22] Gepner D and Qiu Z 1987 *Nucl. Phys.* B **285** 423
[23] Distler J and Qiu Z 1990 *Nucl. Phys.* B **336** 533